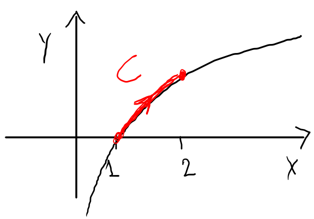
Dú 8

Spočítejte $\int\limits_{\mathcal{C}} x^2\, ds$, kde $\mathcal{C}$ je graf funkce $f(x) = \ln x$, $x \in <1, 2>$.

Řešení:



$$C : \vec{\varphi}(t) = (t, \ln t) \qquad t \in \langle 1, 2 \rangle$$

$$\vec{\varphi}\,'(t) = \left(1, \frac{1}{t}\right)$$

$$\|\vec{\varphi}\,'(t)\| = \sqrt{1 + \frac{1}{t^2}} = \sqrt{\frac{t^2+1}{t^2}} = \frac{\sqrt{t^2+1}}{t} \qquad (t > 0!)$$

$$\int_C x^2\, ds = \int_1^2 t^2 \frac{\sqrt{t^2+1}}{t}\, dt = \int_1^2 t\sqrt{t^2+1}\, dt = \frac{1}{2}\left[(t^2+1)^{3/2} \cdot \frac{2}{3}\right]_1^2 =$$

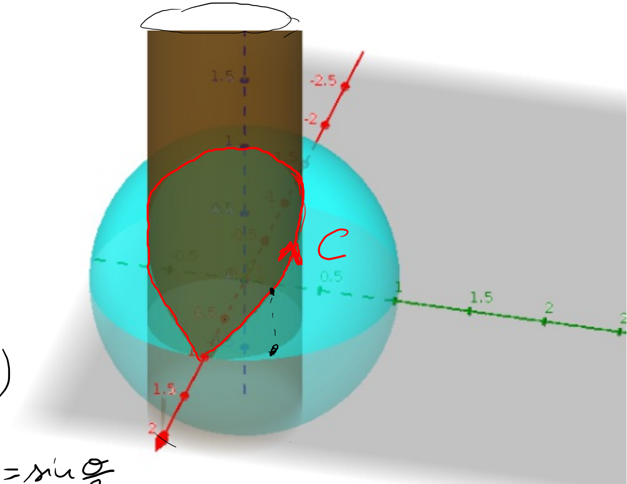
$$= \frac{1}{3}\left(5^{3/2} - 2^{3/2}\right)$$

$$\left[\int\limits_C f\, ds = \int\limits_a^b f(\varphi(t)) \cdot \|\varphi'(t)\|\, dt,\right]$$

Určete $\int_C y^2\,dx + z^2\,dy + x^2\,dz$ kde $C:\quad x^2+y^2+z^2=1 \quad \& \quad x^2+y^2=x \quad \& \quad \underline{\underline{z \geq 0}}$

$(x-1/2)^2+y^2=1/4$

je křivka s kladnou orientací při pohledu shora.



$\varphi(t) = (x(t), y(t), z(t))$

$$\int_C \vec{F}\cdot d\vec{s} = \int_C (\vec{F}\cdot\vec{T})ds = \int_a^b \underline{\vec{F}(\varphi(t))\cdot\varphi'(t)\,dt}\,.$$

Integrál z vektorového pole $\vec{F}$

$\vec{F} = (y^2, z^2, x^2)$

$z^2 = 1-x^2-y^2$

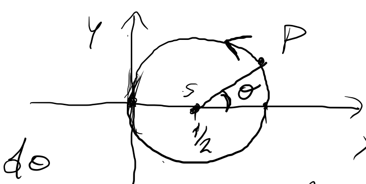
$z = \sqrt{1-x^2-y^2} \qquad 0 \leq \theta \leq 2\pi$

$$\varphi(\theta)\begin{cases} x = \frac{1}{2} + \frac{1}{2}\cos\theta \\ y = \frac{1}{2}\sin\theta \\ z = \sin\frac{\theta}{2} \end{cases}$$

$z = \sqrt{1-\left(\frac{1}{4}+\frac{1}{4}\cos^2\theta+\frac{1}{2}\cos\theta+\frac{1}{4}\sin^2\theta\right)}$

$z = \sqrt{\frac{1}{2}-\frac{1}{2}\cos\theta} = \sqrt{\sin^2\frac{\theta}{2}} = \left|\sin\frac{\theta}{2}\right| = \sin\frac{\theta}{2}$

$\varphi'(\theta) = \left(-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta, \frac{1}{2}\cos\frac{\theta}{2}\right)$



$(x-\frac{1}{2})^2+y^2 = 1/4$

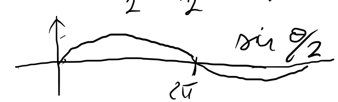
$S(1/2, 0)$

$$\int_C \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\left(\frac{1}{4}\sin^2\theta, \sin^2\frac{\theta}{2}, \left(\frac{1}{2}+\frac{1}{2}\cos\theta\right)^2\right)\cdot\left(-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta, \frac{1}{2}\cos\frac{\theta}{2}\right)d\theta$$

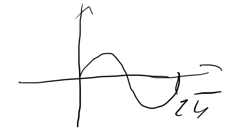
$$= \int_0^{2\pi}\left(-\frac{1}{8}\sin^3\theta + \frac{1}{2}\cos\theta\left(\frac{1}{2}-\frac{1}{2}\cos\theta\right) + \left(\cos^2\frac{\theta}{2}\right)^2\cdot\frac{1}{2}\cos\frac{\theta}{2}\right)d\theta =$$

$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

$\sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$

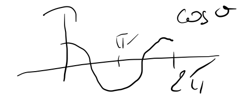


$$= \int_0^{2\pi}\left(\underbrace{\cancel{-\frac{1}{8}\sin^3\theta}}_{0} + \frac{1}{2}\cos\theta\left(\frac{1}{2}-\frac{1}{2}\cos\theta\right) + \left(\cos^2\frac{\theta}{2}\right)^2\cdot\frac{1}{2}\cos\frac{\theta}{2}\right)d\theta =$$

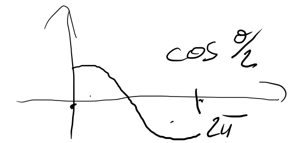


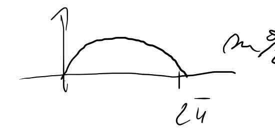
$$= \int_0^{2\pi}\underbrace{\cancel{-\frac{1}{8}\sin^3\theta}}_{0}\,d\theta + \int_0^{2\pi}\underbrace{\cancel{\frac{1}{4}\cos\theta}}_{0}\,d\theta - \frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta + \frac{1}{2}\int_0^{2\pi}\underbrace{\cancel{\cos^5\frac{\theta}{2}}}_{\star}\,d\theta =$$

$\star = 0$



$$= -1/4\int_0^{2\pi}\left(\frac{1}{2}+\underbrace{\cancel{\frac{1}{2}\cos 2\theta}}_{0}\right)d\theta = -\frac{1}{8}\cdot 2\pi = -\pi/4$$





$$\star \int \cos^5 x\,dx = \int \cos x\left(\cos^2 x\right)^2 dx = \int \cos x\left(1-\sin^2 x\right)^2 dx =$$

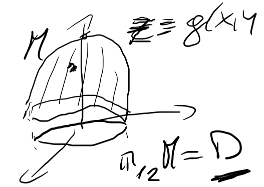
$$= \int \cos x\left(1-2\sin^2 x+\sin^4 x\right)dx = \int\left(\cos x - 2\cos x\,\underbrace{\sin^2 x} + \cos x\,\underbrace{\sin^4 x}\right)dx$$

$u = \sin x$

$du = \cos x\,dx$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5}$$

pro elementární plochu $M = M(g, T)$ *danou* C^1 *funkcí* g *na základní oblasti* T *je*



$$\text{obsah}(M) = S(g,T) = \iint_T \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}. \tag{8.4}$$

$\Phi(x,y) : (x, y, g(x,y))$ $(x,y) \in D$

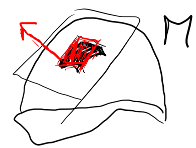
Plocha M může být popsána systémem rovnic $x = \Phi_1(s,t)$, $y = \Phi_2(s,t)$, $z = \Phi_3(s,t)$, $(s,t) \in D$.

Uvedeným rovnicím se říká *parametrizace* elementární plochy M

$$\Phi(s,t) = \big(\Phi_1(s,t), \Phi_2(s,t), \Phi_3(s,t)\big). \qquad (s,t) \in D$$

Věta 8.10. *(Obecná parametrizace) Nechť* $M \subset \mathbb{R}^3$ *je plocha se spojitou parametrizací* $\Phi\colon D \longrightarrow M$, *která je třídy* C^1 *a prostá na vnitřku základní oblasti* $T \subset \mathbb{R}^2$. *Pak*

$$\text{obsah}(M) = \iint_D \left\| \frac{\partial \Phi}{\partial s} \times \frac{\partial \Phi}{\partial t} \right\| dA$$



Integrál z funkce $f : M \to \mathbb{R}$ je určený jako

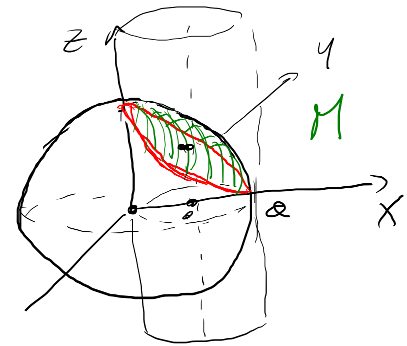
$$\iint_M f \; dS = \iint_U f(\Phi(u,v)) \cdot \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| \; dS \;,$$

kde $\Phi : U \to M$ je vhodná parametrizace.

1 Stanovte obsah části kulové plochy o rovnici $x^2 + y^2 + z^2 = a^2$ (kde $a > 0$ je parametr), kterou z ní vytíná válcová plocha určená podmínkami $x^2 + y^2 = ax$ a $z \geq 0$.

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

$$z^2 = a^2 - x^2 - y^2$$



$$\text{obsah}(M) = \iint_M 1 \, dS$$

$$\phi(x,y) = (\underset{e_1}{x}, \underset{e_2}{y}, \underset{e_3}{\sqrt{a^2-x^2-y^2}}) : (x,y) \in D$$

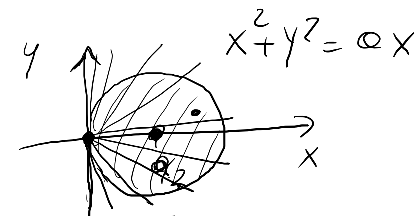
$$\phi_x = \left(1, 0, \frac{-x}{\sqrt{a^2-x^2-y^2}}\right)$$

$$\phi_y = \left(0, 1, \frac{-y}{\sqrt{a^2-x^2-y^2}}\right)$$

$$(\phi_x \times \phi_y) = \left(\frac{x}{\sqrt{a^2-x^2-y^2}}, \frac{y}{\sqrt{a^2-x^2-y^2}}, 1\right)$$

$$\|\phi_x \times \phi_y\| = \sqrt{\frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2} + 1}$$

$$= \sqrt{\frac{a^2}{a^2-x^2-y^2}} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$



$$D = \pi_{12} M$$

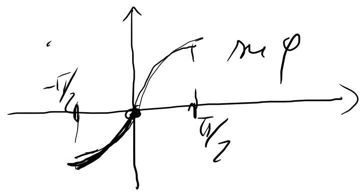
$$\underline{\underline{a > 0}}$$

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi$$
$$\rho^2 = a\rho\cos\varphi$$
$$\underline{\underline{\rho = a\cos\varphi}}$$

$$\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}} \, dA \underset{\text{pd.}}{=} \int_{-\pi/2}^{\pi/2} \int_0^{a\cos\varphi} \frac{-a}{2\sqrt{a^2-\rho^2}} (-2\rho) \, d\rho \, d\varphi =$$

$$= \int_{-\pi/2}^{\pi/2} -a \left[\sqrt{a^2-\rho^2}\right]_0^{a\cos\varphi} d\varphi = \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\varphi} - a\right) d\varphi$$

$$\int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2-a^2\cos^2\varphi}-a\right)d\varphi =$$



$$= \int_{-\pi/2}^{\pi/2}\left(-a\cdot a\underbrace{\sqrt{1-\cos^2\varphi}}_{\sin^2\varphi}+a^2\right)d\varphi = \left[\int_{-\pi/2}^{\pi/2} -a^2\left|\sin\varphi\right|d\varphi\right]+a^2\pi =$$

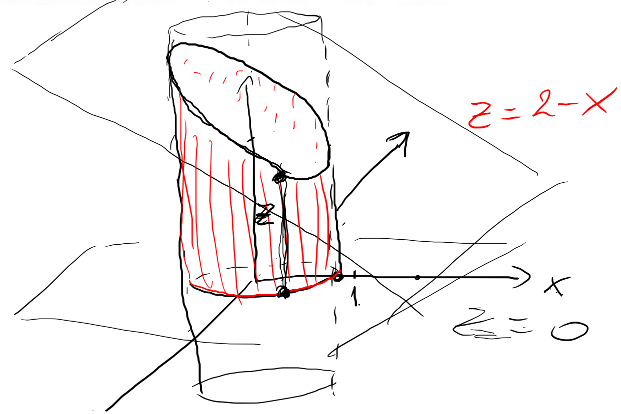
$$= 2\left(\int_{0}^{\pi/2} -a^2\sin\varphi\, d\varphi\right)+a^2\pi = 2a^2\left[\cos\varphi\right]_0^{\pi/2}+a^2\pi =$$

$$= -2a^2+a^2\pi = (\pi-2)a^2$$

$$\sqrt{\sin^2 x} = \left|\sin x\right|$$

$$\int_{-\pi/2}^{\pi/2} a^2\, d\varphi = a^2\int_{-\pi/2}^{\pi/2} 1\, d\varphi = a^2\left[\varphi\right]_{-\pi/2}^{\pi/2} = a^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = a^2\pi$$

2 Spočítejte $\iint\limits_M z\, dS$, kde M je částí válce $x^2+y^2=1$ mezi rovinami $z=0$ a $z=2-x$.

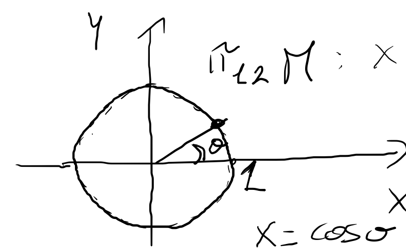


$$M: \phi(\theta, z) = (\cos\theta, \sin\theta, z)$$

$$D \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le z \le 2 - \cos\theta \end{cases}$$

$$\begin{array}{cccc} & e_1 & e_2 & e_3 \\ \phi_\theta = (& -\sin\theta, & \cos\theta, & 0\ ) \end{array}$$

$$\phi_z = (0, \ 0, \ 1) \qquad \phi_\theta \times \phi_z = (\cos\theta, \sin\theta, 0)$$



$\pi_{12} M: x^2+y^2=1$

$x = \cos\theta$

$y = \sin\theta$

$$\phi_\theta \times \phi_z = \det\begin{pmatrix} -\sin\theta & 0 & \vec{e}_1 \\ \cos\theta & 0 & \vec{e}_2 \\ 0 & 1 & \vec{e}_3 \end{pmatrix}$$

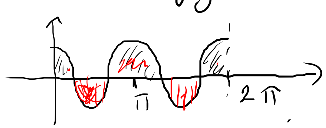
$$\|\phi_\theta \times \phi_z\| = \sqrt{\cos^2\theta + \sin^2\theta + 0} = 1$$

$$\int_0^{2\pi}\int_0^{2-\cos\theta} z \cdot 1 \, dz\, d\theta = \int_0^{2\pi} \left[\frac{z^2}{2}\right]_0^{2-\cos\theta} d\theta =$$

$$= \int_0^{2\pi} \frac{1}{2}\left(4 - 4\underbrace{\cancel{\cos\theta}}_{0} + \cos^2\theta\right) d\theta =$$

$$= 2 \cdot 2\pi + \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2} + \underbrace{\cancel{\frac{\cos 2\theta}{2}}}_{0}\right) d\theta = 2\pi\left(2 + \frac{1}{4}\right) = \frac{9}{2}\pi.$$

$y = \cos 2\theta$



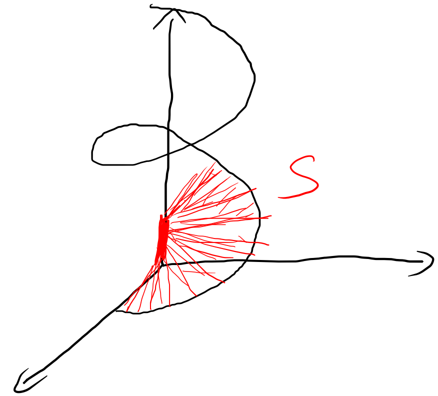
$$\int_0^{2\pi} \cos\theta \, d\theta = \int_0^{2\pi} \sin\theta \, d\theta = 0$$

$\sin\theta$

3 Jaký je celkový náboj na šroubové ploše S dané parametrizací $\Phi(a,b) = (a\cos b,\ a\sin b,\ b)$, $0 \le a \le 1$, $0 \le b \le \pi$, je-li plošná hustota rozložení náboje $\varrho(x,y,z) = \sqrt{1+x^2+y^2}$?

$$\phi(a,b) = (\underset{\vec e_1}{a\cos b},\ \underset{\vec e_2}{a\sin b},\ \underset{\vec e_3}{b})$$

$$\left.\begin{array}{l} a \in \langle 0,1\rangle \\ b \in \langle 0,\pi\rangle \end{array}\right\} D$$



$$\phi_a = (\cos b,\ \sin b,\ 0)$$

$$\phi_b = (-a\sin b,\ a\cos b,\ 1)$$

$$\phi_a \times \phi_b = (\sin b,\ -\cos b,\ \underbrace{a\cos^2 b + a\sin^2 b}_{a})$$

$$\|\phi_a \times \phi_b\| = \sqrt{\sin^2 b + \cos^2 b + a^2} = \sqrt{1+a^2}$$

$$\iint_S \rho(x,y,z)\,dS = \iint_D \sqrt{1+\underbrace{a^2\cos^2 b + a^2\sin^2 b}_{a^2}} \cdot \sqrt{1+a^2}\,dA =$$

$$= \int_0^1 \int_0^\pi (1+a^2)\,db\,da = \pi \int_0^1 (1+a^2)\,da = \pi\left[a + \frac{a^3}{3}\right]_0^1 = \frac{4}{3}\pi$$

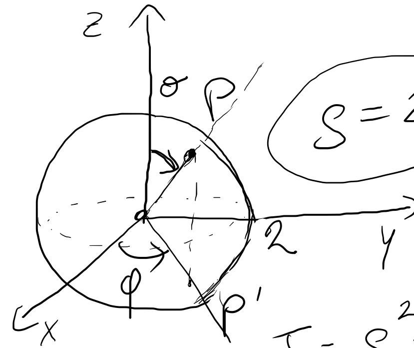
4 Spočítejte $\iint_M x^2 \, dS$, kde M je povrch koule $x^2 + y^2 + z^2 = 4$.

$$M: \quad \phi(\varphi,\theta) = (2\sin\theta\cos\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\theta)$$

Sfer. sou. $\varrho = 2$

$0 \le \theta \le \pi$

$0 \le \varphi \le 2\pi$



$\varrho = 2$

$J = \varrho^2 \sin\theta$

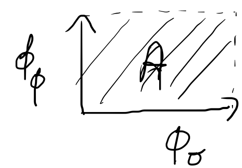
$$\phi_\varphi = (-2\sin\theta\sin\varphi,\ 2\sin\theta\cos\varphi,\ 0)$$

$$\phi_\theta = (2\cos\theta\cos\varphi,\ 2\cos\theta\sin\varphi,\ -2\sin\theta)$$

$\phi_\varphi \perp \phi_\theta$

$$\phi_\varphi \cdot \phi_\theta = -4\sin\theta\cos\theta\sin\varphi\cos\varphi + 4\sin\theta\cos\theta\sin\varphi\cos\varphi + 0 = 0$$

$$\|\phi_\varphi \times \phi_\theta\| = 4\sin\theta$$



$\|\phi_\varphi \times \phi_\theta\| = \text{obsah } A$

$= \|\phi_\varphi\| \cdot \|\phi_\theta\|$

$$\iint_M x^2 \, dS = \int_0^{2\pi}\int_0^{\pi} 4\sin^2\theta\cos^2\varphi \cdot 4\sin\theta \, d\theta \, d\varphi =$$

$$= 16\int_0^{2\pi}\cos^2\varphi \, d\varphi \cdot \int_0^{\pi}\sin^3\theta \, d\theta =$$

$$= 16\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cancel{\cos 2\varphi}\right) d\varphi \cdot \int_0^{\pi}\sin\theta(1-\cos^2\theta)\, d\theta =$$

$$= 16 \cdot \frac{1}{2} \cdot 2\pi \cdot \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi} = 16\pi\left(2 - \frac{2}{3}\right) = \frac{64}{3}\pi$$

5 Spočítejte $\iint\limits_M (10 - z)\ dS$, kde $M = \{(x, y, z) : z = \sqrt{x^2 + y^2}, 1 \leq z \leq 4\}$.

Připomenutí: Tok vektorového pole $\vec{F} : M \to \mathbb{R}^3$ orientovanou plochou $M \subseteq \mathbb{R}^3$ se spočítá jako

$$\iint_M \vec{F} \cdot d\vec{S} = \iint_U \vec{F}\big(\Phi(u,v)\big) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) dA \ ,$$

kde $\Phi : U \to M$ je opět vhodná parametrizace, $U \subseteq \mathbb{R}^2$, a orientace daná vektorovým polem $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}$ souhlasí se zadanou parametrizací plochy M. (Pokud by orientace nesouhlasila, stačí jen změnit pořadí ve vektorovém součinu, tj. změnit znaménko integrálu.)

6 Spočítejte $\iint_M \vec{F} \cdot d\vec{S}$ kde $\vec{F}(x,y,z) = (0, y, -z)$ a M je část paraboloidu $y = x^2 + z^2$ pro $y \leq 1$ s orientací danou vektorovým polem směřujícím do leva ($y \leq 0$).

7 Kapalina s hustotou 1 protéká s rychlostí danou polem $\vec{v}(x,y,z) = (y,\ 1,\ z)$. Určete průtok kapaliny směrem vzhůru plochou S, která je částí paraboloidu $z = 9 - \frac{(x^2+y^2)}{4}$ pro $x^2 + y^2 \leq 36$ (neboli určete $\iint\limits_S \vec{v} \cdot d\vec{S}$).

8 Spočítejte $\iint\limits_M \vec{F}\cdot d\vec{S}$, kde $\vec{F}(x,y,z)=(x,xy,xz)$, a M je část roviny $3x+2y+z=6$, která leží v prvním oktantu a je orientovaná směrem vzhůru.